TURING DEGREES

AND

RANDOMNESS FOR CONTINUOUS MEASURES

JAN REIMANN

joint work with Mingyang Li

Randomness with respect to μ

There exists a representation R_μ of μ s.t. the real passes all μ-ML-tests that have access to R_μ

R & Slaman (2015) A real x is random for some μ with $\mu(\{x\})=0$ iff x is not computable

Things get more interesting if we pass to

- stricter randomness notions (Haken, 2014)
- smaller families of measures

this talk: continuous measures

R & Slaman (2015) If x is not Δ^1_1, then it is random for a continuous measure

NCR = reals never random for a contin measure

What is the structure of NCR?

R (2008) If $h : \mathbb{N} \to [0,\infty)$ is non-decreasing, unbounded, and computable, and

$$-\log \underset{\sim}{M}(X \upharpoonright n) \geq h(n),$$

then X is random for a continuous measure

This implies $\{X\}$ is not effectively $\mathcal{H}^h$-null

Effective Frostman Lemma

For a measure $\mu \in \mathcal{P}(2^{\mathbb{N}})$, define its

dissipation function $\quad g_\mu(n) = \min \{ \ell \ \forall |\sigma| = \ell \ \ \mu(\sigma) < 2^{-n} \}$

Frostman (1935) For A analytic, if $\dim_H A > s$, then there exists a measure μ st

$$\operatorname{Supp}(\mu) \subseteq A \qquad \& \qquad g_\mu \in O\left(\frac{n}{s}\right)$$

What is the structure of NCR inside Δ^1_1?

Kjos-Hanssen & Montalban (2005)

If x is a member of a countable Π^0_1 class, then $x \in$ NCR

$\longrightarrow$ NCR is cofinal in the Turing degrees of Δ^1_1

(Cenzer, Clote, Smith, Soare, Wainer)

NCR is a Π^1_1 class and hence has a Π^1_1 rank

The Kjos-Hanssen - Montalban result suggested this could be the Cantor-Bendixson rank

x Δ^0_2 with recursive approximation $\gamma(n,s)$

settling function

$$c_\gamma(n) = \min\{s \ \ \forall t \geq s \ \ \gamma(n,t) = \gamma(n,s)\}$$

R & Slaman If x is Δ^0_2 and μ-random, then

$c_\gamma(n) > g_\mu(n)$ for all but finitely many n

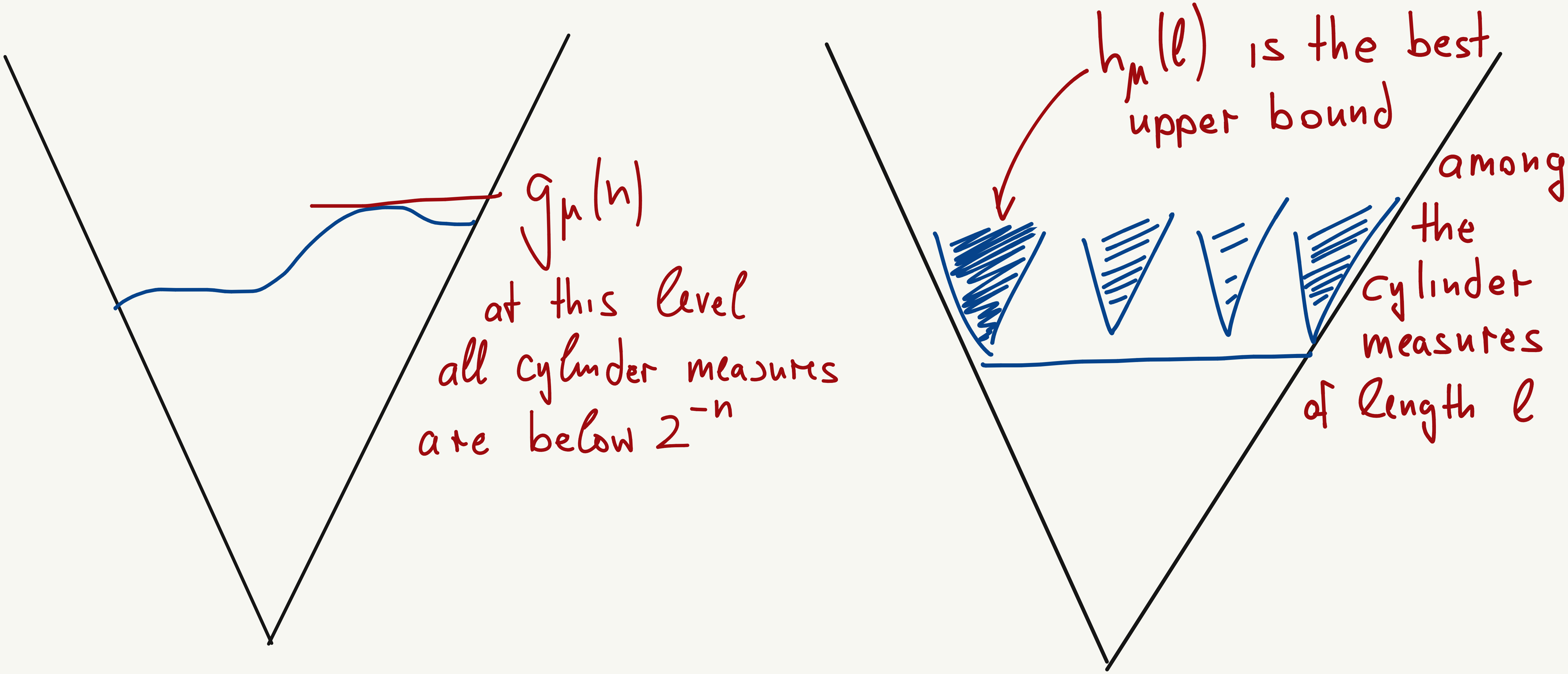
Barmpalias, Greenberg, Montalban & Slaman used this to show

Every x Turing below an incomplete r.e. degree is in NCR

(In particular, all K-trivials are in NCR)

The granularity function of a measure μ

$$h_\mu(\ell) = \max\{n \quad \forall |\sigma| = \ell \quad \mu(\sigma) < 2^{-n+1}\}$$



We have

$$n < g(n) < g(n+1) < g(g(n+1))$$

$$h(\ell) < h(\ell+1) \leq h(\ell)+1 \leq \ell+1$$

$$h(g(n)) = n+1$$

$$h(\ell) \longrightarrow \infty$$

$$g \equiv_T h$$

Let μ be a continuous measure

A level-n Solovay test for μ is a μ-r.e. set of strings $\{\sigma_i \; i \in \mathbb{N}\}$ s.t. $n \geq 1$

$$\sum_i \left(h^{(n)}(|\sigma_i|)\right)^{\log n} 2^{-h^{(n)}(|\sigma_i|)} < \infty$$

$h^{(n)}$ n-th iterate of h

$x \in 2^{\mathbb{N}}$ is non-μ-random of level n if it fails some level-n Solovay test, i.e. $x \in [\sigma_i]$ for infinitely many $i \in \mathbb{N}$

level ω non-random of level n f all $n \geq 1$

If $x \leq_T \mu$, then x is non-μ-random of level ω

If x is non-μ-random of level 1, then x is not μ-ML-random

Every level-$(n+1)$ test is also a level-n test

NCR of level n non-μ-random of level n for all continuous μ

NCR^n

$$NCR^\omega \Rightarrow NCR^{n+1} \underset{\overset{?}{\not\Leftarrow}}{\Rightarrow} NCR^n \Rightarrow NCR$$

$f\ \mathbb{N} \to \mathbb{N}$ is a modulus for $x \in 2^{\mathbb{N}}$

if every function that dominates f

computes x

self-modulus $\quad f \equiv_T x$

Every $x \leq_T 0'$ has a self-modulus

<u>Construction 1</u> given $X = x_0 x_1 x_2 \quad \in 2^{\mathbb{N}}$

$f \equiv_T X$ self-modulus

$$y_0 = 1^{f(0)} \frown 0 \frown x_0$$

$$y_{n+1} = y_n \frown 1^{f(|y_n|)} \frown 0 \frown x_{n+1}$$

Put $y = \lim_n y_n$ Then $y \equiv_T X$

THM If x has a self-modulus f and y is defined from x as in Construction 1, then $y \in NCR^{\omega}$

LEM If f is not dominated by any μ-computable function, then $\exists^{\infty} n \quad g_{\mu}^{*(k)}(2|y_n|+1) < f(|y_n|)$

μ-computable variant of g_{μ}

Solovay test of level k

$$T_k = \{ \sigma ^\frown 1^{g_{\mu}^{*(k)}(2|\sigma|)} \quad \sigma \in 2^{<\mathbb{N}} \}$$

Construction 2 Given $x \in 2^{\mathbb{N}}$, y r.e. above x

Assume $y = W_e^x$

Let $m_i = \min\{ j > i \quad \Phi_e^x(j)\downarrow \}$

$$f(i) = \begin{cases} \min\{ s \quad \forall k \le m_i \ (\Phi_e^x(k)\downarrow \implies \Phi_{e,s}^x(k)\downarrow)\} & \text{if } i \in y \\ 1 & \text{if } i \notin y \end{cases}$$

$f \le_T y$

$$z = y_0^{f(0)} \frown 0 \frown y_1^{f(1)} \frown 0 \frown y_2^{f(2)} \frown 0 \frown$$

$z \equiv_T y$

THM For any continuous μ, if x is non-μ-random of level $2n$ and y is re a x, and z is obtained from y via Construction 2 then z is non-μ-random of level n

COR For all n, every n-re degree contains an NCR (of level ω) element

A, B simultaneously cont random

$\exists Z, \mu$ cont , $Z \geq_T \mu$ s t

A, B μ-random rel to Z

Conjecture (Day & Marks)

A, B NSCR $\Longleftrightarrow$ A or B NCR

f self-modulus for $\emptyset'$

$$S_0 = \emptyset, \quad S_{n+1} = \{\sigma^\frown 1^{f(|\sigma|)\frown} 0^\frown a \;\; \sigma \in S_n, \; a \in \{0,1\}\}$$

$$S = \{y \in 2^{\mathbb{N}} \;\; \forall n \, \exists \sigma \in S_n \;\; \sigma \sqsubset y\}$$

Then

- S is perfect
- If $y \in S$ is μ-random, then any representation of μ computes a function dominating f

Pick x_1 ML-random $\in \mathring{\Delta}_2$

Distribute unit mass uniformly along S

$\rightsquigarrow$ cont μ Pick x_2 μ-random

Then $x_1, x_2 \notin$ NCR, but

$$(x_1, x_2) \in \text{NSCR}$$